Rigidity percolation

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Rigidity

- Consider bars, which have a fixed length, linked together by "joints". Is the system rigid or floppy?

Example in 2 dimensions; bar lengths are fixed, not the angles:

- Floppy
- Rigid, not overconstrained
- Rigid, overconstrained
Rigidity

- When there are only a few joints and bars, it is easy...
  What about this network, with 11 sites?

- Is it floppy? Rigid? How many floppy modes? Where?
Related problems

- Bond bending constraints: angles between two adjacent bonds have to be kept fixed ($D = 3$)

\[ N_{\text{deform}} = 1 \]
- Bond bending constraints: angles between two adjacent bonds have to be kept fixed \((D = 3)\)
- Rigidity with "sliders": some joints constrained to move on a line
Related problems

• Bond bending constraints: angles between two adjacent bonds have to be kept fixed ($D = 3$)
• Rigidity with "sliders": some joints constrained to move on a line
• Rigidity with "pinned" joints, which cannot move at all
An application: "covalent glasses"

- Example: a disordered network with Germanium and Selenium atoms. Ge = 4 bonds; Se = 2 bonds.
- Bond lengths and angles between two adjacent bonds can be considered as constraints (∼the energy needed to modify them is larger than the temperature).

Each bond: 1 length constraint; each Se atom: 1 angular constraint; each Ge atom: 5 angular constraints.

→ Go from "floppy" to "rigid" by increasing the Ge fraction.
Another application: protein rigidity (MF Thorpe and coworkers)

- Proteins are large biological molecules. An example (hexokinase):

Let’s simplify:
Atoms \(\rightarrow\) balls; chemical (or other strong) bonds \(\rightarrow\) bonds; weak interactions \(\rightarrow\) forgotten!

\(\rightarrow\) is the simplified structure floppy or rigid?
\(\rightarrow\) if floppy, what are the possible deformations?
Constraint counting

Maxwell’s idea: constraint counting
• each joint starts with 2 degrees of freedom
• each bar removes one degree of freedom
→ First try: formula for the number of remaining degrees of freedom, \( N_{d.o.f.} \); \( N \) joints, \( M \) bars:

\[ N_{d.o.f.} = 2N - M \text{ if } M < 2N - 3 \; ; \; N_{d.o.f.} = 3 \text{ if } M \geq 2N - 3 \]

• Cannot be correct... Need to count redundant constraints:

\[ N_{d.o.f.} = 2N - M + N_{\text{redundant}} \]

\( N = 5; M = 7 \)

\( N_{\text{redundant}} = 1 \)

\( N_{d.o.f.} = 4 \)
From geometry to graph theory: Laman theorem

• Power of constraint counting: replace a geometrical problem by a discrete, graph theoretical one.

**Question:** is it possible to keep this desirable feature, correcting the approximations of constraint counting?

• **Generic rigidity** in 2D can be characterized in a purely graph theoretical way (Laman 1970):

  \[ G \text{ has a redundant constraint } \iff \text{ there is a subgraph with } n \text{ vertices, } m \text{ edges and } m > 2n - 3. \]

  \[ \rightarrow \sim \text{ constraint counting on each subgraph to detect redundant constraints} \]
Generic rigidity

Top: a non generic realization; Laman theorem does not apply. Bottom: a generic realization of the same graph.
Second ingredient: probabilities

In many cases, the structure is too large to be known exactly (think of covalent glasses for instance) → one would like to use a probabilistic description

Each link between a pair of neighboring vertices is present with proba. $p < 1$

Links put "randomly", no geometry.

It is a percolation problem.
"Standard" percolation

- "connectivity" percolation = well studied since the 60's

- Each link is present with proba. p, and absent with proba. 1-p
- Question: is there a path from top to bottom?

NB1: standard percolation is analog to "rigidity" percolation with one "degree of freedom" per vertex
NB2: standard percolation on a random graph = appearance of a "giant connected component"
Erdos-Renyi random graphs

Definition of $G(n, c/n)$: $n$ vertices; any pair of vertices connected with proba. $c/n$. There is no notion of space.

Some properties: approximately $nc/2$ edges; Poisson $\mathcal{P}(c)$ degree distribution; few small loops...
Questions for rigidity percolation

- Is there a well defined threshold $p_c$ for the appearance of a "macroscopic rigid cluster"?

\[ p < p_c \Rightarrow \text{percolation probability} = 0 \]
\[ p > p_c \Rightarrow \text{percolation probability} = 1 \]

**Answer**: yes for random graphs and lattices (Numerics in the 90’s; Holroyd ~2000); threshold computed by Kasiwisvanathan, Moore and Theran (KMT 2011) for $G(n, c/n)$ random graphs, unknown for lattices.
Questions for rigidity percolation

- Size of the largest rigid component? Continuous/discontinuous at $p_c$?

![Size of the largest rigid component](image)

Size of the largest rigid component

continuous

discontinuous
Questions for rigidity percolation

- Size of the largest rigid component? Continuous/discontinuous at $p_c$?

**Answer:**
- Discontinuous for $G(n, c/n)$ random graphs (Theran)
- Seems to be continuous for lattices (Jacobs-Thorpe, Duxbury-Moukarzel 90’s, numerics).
Questions for rigidity percolation

- Size of the largest rigid component? Continuous/discontinuous at $p_c$?
  Example: Erdős-Rényi random graph $G(n, c/n)$. Vary $c$

Size of the biggest rigid and stressed clusters, and number of "floppy modes" vs mean connectivity
Questions for rigidity percolation

- For lattices, what happens close to threshold? = ”Critical” behavior? $\beta = ?$ (critical exponent, exciting for statistical physicists); fractal dimension?

Overconstrained regions (Simulation by P. Duxbury et al.)

**Answer**: unknown. Critical exponents seem to be different from standard percolation.
Goals

Fully understand the 2D lattice case: difficult... More modest goals:

1. Find models that can be solved;
2. Explore similarities/differences standard percolation/rigidity percolation: study models that interpolate between both.

→ Study rigidity percolation with sliders on random graphs

→ Study other kind of "simple" lattices (eg. hierarchical).
Rigidity with sliders

- Consider a structure with \( n_1 \) sites with sliders, \( n_2 \) free sites and \( m \) bars. One slider = one constraint
- modify constraint counting

Difficulty: sliders ”pin” the rigid components to the plane
Distinguish between free, partly pinned, and pinned rigid clusters

A Laman-type theorem (I. Streinu, L. Theran, 2010).
Redundant constraint \( \iff \) subgraph with

\[ n'_1 + 2n'_2 - m' - \max(3 - n'_1, 0) < 0 \]

\( \rightarrow \) A graph theoretical approach possible (under a genericity condition, as usual)
Rigidity percolation with sliders

- Erdős-Renyi random graph $G(n, c/n)$, with $n = n_1 + n_2$
  $n_1 = (1 - q)n$, $n_2 = qn$.
  $1 - q =$ proportion of sites with sliders

- $q = 0$: ordinary percolation = well known; continuous

- $q = 1$: rigidity percolation, discontinuous; threshold $c = 3.588\ldots$

- What happens in between?
Threshold

- percolation threshold vs proportion of sliders

\[ c^* = \frac{1}{1 - q} \] for \( q \leq \frac{1}{2} \)

- For \( q > \frac{1}{2} \), implicit expression for \( c^*(q) \):

\[
c^* = \frac{\xi^*}{1 - e^{-\xi^*} - q\xi^*e^{-\xi^*}}, \quad \frac{\xi^*(1 - e^{-\xi^*} - q\xi^*e^{-\xi^*})}{(1 + q)(1 - e^{-\xi^*} - q\xi^*e^{-\xi^*}) - q(\xi^*)^2e^{-\xi^*}} = 2.
\]
Theorem: (JB, M. Lelarge, D. Mitsche)

Let $G \sim \mathcal{G}(n, c/n)$ an Erdos-Renyi random graph, with a fraction $1 - q$ of sliders. Then, we can compute $c^*(q)$, such that with high probability (proba $\to 1$ when $n \to \infty$):

- If $c < c^*(q)$, there is no giant rigid component
- If $c > c^*(q)$, there is a giant rigid component

Furthermore, for $q < 1/2$ the transition is continuous, and for $q > 1/2$ it is discontinuous.

NB: $c^*(q = 0) = 1$ and $c^*(q = 1) = 3.588\ldots$
Size of the largest rigid component

- Size of the largest component at threshold: jump for $q > 1/2$: $\sim$ rigidity without sliders.
- Continuous transition for $q < 1/2$: $\sim$ connectivity percolation.
- Discontinuous transition for $q > 1/2$.

$\rightarrow$ *tricritical point* at $q = 1/2$ (statistical mechanics jargon)
Strategy of proof

- **Step 1:** Link with *orientability* (generalizes the case without sliders)

  - Intuition: one bond removes one degree of freedom to one of the two vertices it links
  - Vertices with or without slider: 1 or 2 degree of freedom
  → Link with "orientability"

![Diagram](image-url)
Strategy of proof, 2

- **Step 2:** Thresholds for orientability and percolation are equal
  "Rigid" $\Rightarrow$ "Non orientable" = easy
  "Non orientable" $\Rightarrow$ "Rigid" = more laborious

- **Step 3:** Compute the threshold for orientability $\rightarrow$ method introduced by M. Lelarge
  $\sim$ rigorous "cavity method", a heuristic introduced by physicists.
**Step 4: Type of transition**

- For $q > 1/2$ ("rigidity-like" transition), a density argument applies: rigid components must be dense enough, and dense subgraphs must have a minimal size of order $n$ (uses again the generalization of L. Theran’s lemma).
  \[ \rightarrow \text{discontinuous transition} \]

- For $q < 1/2$ ("connectivity-like" transition), we need "cores"

Remove recursively blue sites with less than 2 links and red sites with less than 3. What remains is the "2.5-core". Then add recursively blue sites with one link to the core, and red sites with 2. One gets the "2.5 + 1.5-core".
Step 4: type of transition

- For $q > 1/2$ ("rigidity-like" transition), a density argument applies: rigid components must be dense enough, and dense subgraphs must have a minimal size of order $n$ (uses again the generalization of L. Theran’s lemma).
  → discontinuous transition

- For $q < 1/2$ ("connectivity-like" transition), we need "cores"

Then show: largest rigid component $\subset 2.5 + 1.5$-core

- Compute the size of the $2.5 + 1.5$-core and show it is small.
Step 5: Size of cores

• Size of the $3 + 2$ core = conjecture in Kasivisvanathan-Moore-Theran 2011.

• Strategy: use Janson-Luczak technique

Bins = vertices, with sliders (blue) or without (red)
Balls = half edges

→ good knowledge of degree distributions after the core construction
→ possible to control the process growing the $3+2$ core.
Conclusions on random graphs

- Complete phase diagram with a tricritical point
- Proof combines many "old" ideas: strategy Theran et al. relating to orientability; M. Lelarge's technique to compute orientability threshold; Janson-Luczak technique to compute the size of cores
- What about rigidity with some pinned sites? Conjecture by physicists (Moukarzel '03): the discontinuous transition may disappear, but there is no continuous transition... A proof seems accessible - joint work with Dieter Mitsche and Louis Theran
- Physics literature: tree-like heuristics give access to much more detailed results (Large Deviation Cavity Method); could these be transformed into theorems? A general question, beyond rigidity.
Beyond random graphs?

- Random graphs: much easier than percolation problems on lattices . . .
- whereas problems on lattices, or at least on graphs with some geometric content, are a priori more interesting for physics.
- Understand the phase transition on regular lattices (beyond existence proof by Holroyd)? Precise numerical simulations would be useful; I don’t even have heuristic theoretical ideas... → a lot to do here!