Random tiling dynamics

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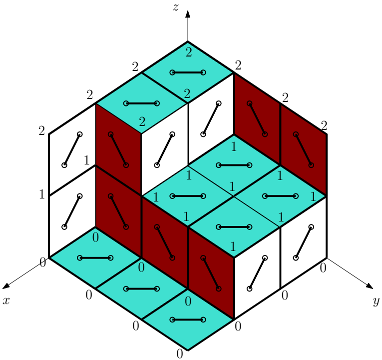
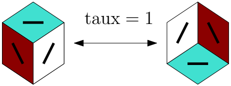
Glauber dynamics of random planar tilings

Markov chain on tilings/perfect matchings



Elementary updates:



Glauber dynamics of random planar tilings

Soft facts:

- Ergodicity;
- Unique invariant measure: uniform π_Λ (also reversible)
- finite mixing time:

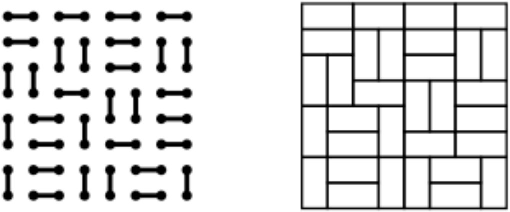
$$T_{\rm mix} = T_{\rm mix}(\Lambda) = \{\inf t > 0 : \max_\eta \|\mu_t^\eta - \pi_\Lambda\|_{TV} \le 1/4\}$$

- Dynamics is monotone: stochastic order is preserved.

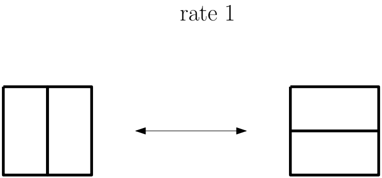
Questions:

- Rapid mixing? ($T_{\rm mix}$ polynomial in $|\Lambda|$)
- Precise estimates on $T_{\rm mix}$ as Λ large?
- What "typical path" for convergence to equilibrium?

Another example: domino tilings

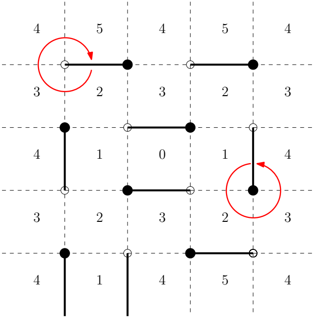


Elementary updates:



Another example: domino tilings

Graph is planar and bipartite ⇒ height function



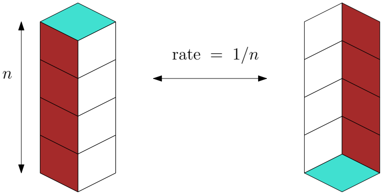
Dimers ⟺ Height gradients

Comments

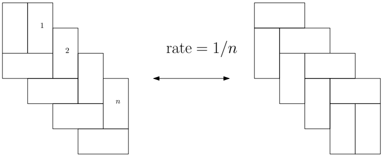
- Non-obvious fact: counting number of configurations is computationally easy (compute $|\Lambda| \times |\Lambda|$ determinant).
- As a consequence, $\exists$ ways of sampling random rhombus- or domino tilings that are quick (even quicker than Glauber dynamics (Mucha-Sankowski, D. Wilson, ...)).
- Appeal of Glauber dynamics for statistical physics: Markov dynamics of discrete interface (height function $h(x,t)$).
- Law of large numbers for $h(\cdot,t)$ suitably rescaled as $L^{-1}h(L\cdot,L^2t)$? (Diffusive scaling. Here, L is diameter of Λ)

The LRS "tower-move" dynamics

Luby-Randall-Sinclair '01: auxiliary dynamics with "tower-moves"



Similar for dominos:



The LRS "tower-move" dynamics

- Easy: ergodicity, invariance & reversibility of π_Λ, monotonicity of dynamics are still ok
- Non-trivial fact: mutual volume between configurations

$$\Delta V(h^{(1)}(t), h^{(2)}(t)) := \sum_x h^{(1)}(x,t) - h^{(2)}(x,t)$$

  is super-martingale (martingale apart from boundary effects)
- easy to deduce via a coupling argument:

$$T_{\text{mix}}^{tower} = O(L^{6+o(1)}) \quad \text{(Rapid mixing)}$$

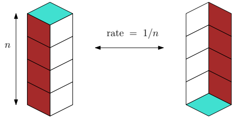
- idea:
  - enough to estimate coalescence time of maximal and minimal configurations
  - mutual volume is at most $O(L^3)$
  - symmetric RW in $[0,\dots,L^3]$ hits zero in time at most $L^{2\times 3+o(1)}$.

The LRS "tower-move" dynamics

- Simple to deduce (via comparison of Markov chain spectral gaps):

$$T_{\mathrm{mix}} = O(L^{8+o(1)})$$

where $8 = 6 + 2$, because tower moves have size at most L and $n^{2+o(1)}$ single-flip steps are needed to simulate a size-n jump.

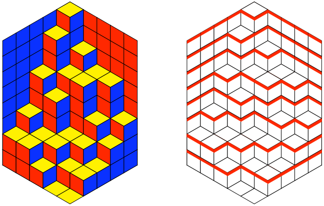


- More refined result by D. Wilson '04:

$$T_{\mathrm{mix}}^{tower} = O(L^2 \log L)$$

(optimal) for rhombus tiling "tower" dynamics.

Wilson's idea

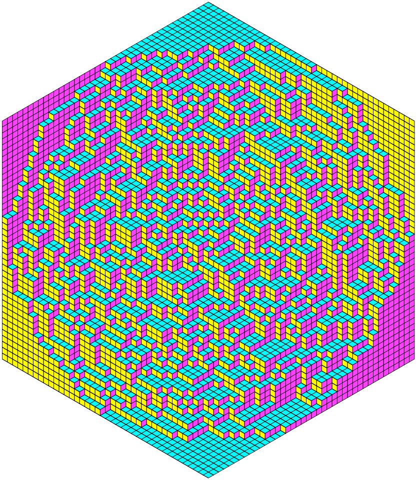


$$\partial_t \mathbb{E} H(x,t) = \Delta_x \mathbb{E} H(x,t), \quad H(x,t) = \sum_{j=1}^{L} p^{(j)}(x,t), \quad -L \le x \le L.$$

The $L^2 \log L$ mixing time bound comes from heat equation scaling:

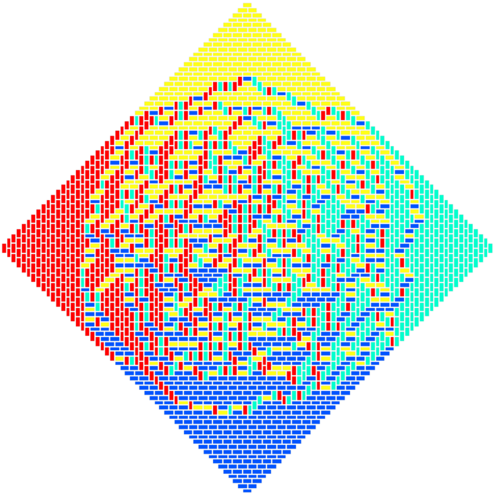
$$\mathbb{E}\Delta V(\eta^+(t), \eta^-(t)) \approx L^3 e^{-t/L^2} \ll 1 \qquad \text{if} \qquad t \ge cL^2 \log L.$$

Macroscopic shapes



Size-L hexagon Λ_L

Macroscopic shapes



Size-L square Λ_L

Macroscopic shapes

In both cases, $L^{-1}\Lambda_L$ tends as $L \to \infty$ to domain $D \subset \mathbb{R}^2$ and the boundary height tends to limit function on ∂D.

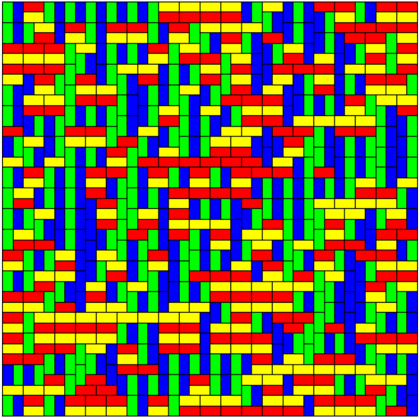
Theorem [Cohn-Kenyon-Propp '01] There exists deterministic function $\bar{h} : D \mapsto \mathbb{R}$ such that

$$\lim_{L\to\infty} \pi_{\Lambda_L} \left(\left| \frac{1}{L} h(xL) - \bar{h}(x) \right| \geq \epsilon \right) = 0$$

for every $x \in D$.

The marcroscopic shape can be C^∞ or exhibit “frozen regions”

Macroscopic shapes



In this case, $\bar{h}(\cdot)$ is affine.

An almost-optimal mixing time result

Theorem 1 [P. Caputo, F. Martinelli, F.T., '12, B. Laslier, F.T., '15] Assume that the macroscopic shape $\bar{h}(\cdot)$ is affine. Then,

$$T_{\mathrm{mix}}(\Lambda_L) = O(L^{2+o(1)}).$$

Theorem 2 [B. Laslier, F.T., '15] Assume that $\bar{h}(\cdot)$ is C^∞. Then, at time $t \geq L^{2+\epsilon}$, w.h.p.

$$\left| \frac{1}{L} h(x,t) - \bar{h}(x/L) \right| \leq \delta \quad \forall x \in \Lambda.$$

(does not imply T_{mix} upper bound)

Note: single-flip and "tower" dynamics are essentially equally fast.

Comments on the result

Proof is kind of involved.

Main ingredients:

1. under π_{Λ_L}, height fluctuations from macroscopic shape are w.h.p. $O(\log L)$
2. item (1) plus Wilson's result on T_{mix}^{tower} gives: if initial height is $L^{o(1)}$ away from equilibrium profile, equilibrium is reached after time $L^{2+o(1)}$
3. Peres-Winkler "censoring inequality" for monotone Markov chains

Expected: Hydrodynamic limit

We expect: if the initial condition approximates smooth profile,

$$\lim_L \frac{1}{L} h(xL) = \phi_0(x)$$

then

$$\lim_L \frac{1}{L} h(xL, tL^2) = \phi(x,t)$$

with ϕ solving parabolic, non-linear PDE

$$\partial_t \phi = \mu(\nabla\phi) \sum_{i,j=1}^{2} \sigma_{i,j}(\nabla\phi)\partial^2_{x_i,x_j}\phi.$$

$\{\sigma_{i,j}\}$: positive symmetric matrix, Hessian of entropy function.
$\mu(\cdot) > 0$: mobility.

Hydrodynamic equation can be rewritten as

$$\partial_t \phi = -\mu(\nabla\phi)\frac{\delta\Sigma[\phi]}{\delta\phi(x,t)}$$

with $\Sigma[\phi]$ the entropy functional

$$\Sigma[\phi] = \int \sigma(\nabla\phi)dx.$$

Entropy: indepent of transition rates

$$-\sigma(\rho) = \lim_{L\to\infty}\frac{1}{L^2}\log\#\{\text{tilings of } L\times L \text{ domain with global slope } \rho\}$$

Mobility μ: depends on the Markov chain rates

Linear response

In general (e.g. single-flip Glauber) not possible to compute μ explicitly.
Green-Kubo formula (non-rigorous, linear response theory):

$$\begin{aligned} \mu_{GK}(\rho) &= \lim_{L} \frac{1}{2L^2} \pi_{L,\rho} \left[\sum_{flips} c_{flip}(h)[\Delta V(flip)]^2 \right] \\ &\quad - \lim_{L} \frac{1}{L^2} \int_0^\infty \mathbb{E}_{\pi_{L,\rho}} [\mathrm{Drift}(h(0))\, \mathrm{Drift}(h(t))]\, dt \end{aligned}$$

$\pi_{L,\rho}$: uniform measure on $L \times L$ torus, restricted to configurations with slope ρ.

$$\mathrm{Drift}(h) = \sum_{flip} c_{flip}(h) \Delta V(flip)$$

Gradient condition

It may happen that summation by parts on the torus gives

$$\mathrm{Drift}(h) \equiv 0 \qquad (\star)$$

for every configuration h.

- For single-flip Glauber dynamics, $(\star)$ does not hold.
- For tower-move dynamics, it does.
  Origin: martingale property of volume
- Moreover, in this case

$$\mu_{GK}(\rho) = \lim_{L} \frac{1}{2L^2} \pi_{L,\rho} \left[\sum_{flips} c_{flip}(h) [\Delta V(flip)]^2 \right]$$

can be computed [S. Chhita, P. Ferrari '15, B. Laslier, F.T., '17]

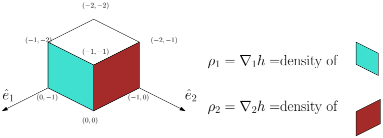
Green-Kubo mobility for the tower dynamics

Infinite-volume measure $\pi_\rho = \lim_L \pi_{L,\rho}$ has determinantal structure:

$$\pi_\rho(\text{event involving } n \text{ edges}) = \det(n \times n \text{ matrix}).$$

Computation gives:

$$\mu_{GK}(\rho) = \frac{1}{\pi} \frac{\sin(\pi\rho_1)\sin(\pi\rho_2)}{\sin(\pi(1-\rho_1-\rho_2))}$$



A hydrodynamic limit for the tower dynamics

We need two assumptions:

- We work with periodic b.c. (dynamics on the torus).
- The initial profile ϕ_0 is smooth (say C^2, because we were lazy) and nowhere "extremal" (cannot be weakened).

Theorem 3 [B. Laslier, F. T. '18]
For every $t > 0, x \in [0,1]^2$, convergence to the limit PDE:

$$\mathbb{P}\left(\left|\frac{h(xL, tL^2)}{L} - \phi(x,t)\right|\right) > \epsilon \to 0$$

with ϕ unique, classical solution of

$$\partial_t \phi = \mu_{GK}(\nabla\phi) \sum_{i,j=1}^{2} \sigma_{i,j}(\nabla\phi)\partial^2_{x_i,x_j}\phi.$$

Remarks on the PDE

- non-trivial fact I: $\mathbb{L}^1$ contraction:

$$\partial_t \int_{[0,1]^2} dx(\phi^{(1)}(x,t) - \phi^{(2)}(x,t)) = 0.$$

  Microscopic origin: volume between 2 configurations is martingale

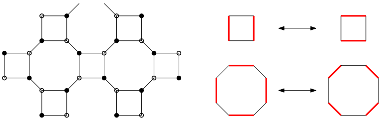
- non-trivial fact II: $\mathbb{L}^2$ contraction.

$$\partial_t \int_{[0,1]^2} (\phi^{(1)}(x,t) - \phi^{(2)}(x,t))^2 \leq 0$$

  (would be trivial if $\mu_{GK}(\cdot) \equiv 1$).

Open problems

- Prove (or disprove) that T_{mix} of Glauber dynamics in size-L hexagon or size-L Aztec diamond is $O(L^{2+o(1)})$. Issue: frozen regions. Best upper bound: $O(L^{4+o(1)})$.
- Rapid mixing for perfect matchings of more general planar bipartite graphs?
  E.g.

  Missing: generalization of tower-move trick.
- Possible slow mixing due to "gaseous phases"??

Conclusions

- single-flip version of the process is very hard, only bounds on mixing/relaxation time...
- ...but “natural” modified version (tower-dynamics) can be analyzed in detail (T_{mix}, law of large numbers for height profile)
- Ongoing project (with T. Funaki): dynamical large deviations. Explicit LDP functional

$$\frac{1}{L^2} \log \mathbb{P}(L^{-1} h(L\cdot, L^2\cdot) \sim \phi(\cdot,\cdot)) \stackrel{L\to\infty}{\to} \mathcal{I}(\phi)$$

Thanks!